Topic: Complex Numbers
Target Age: 16 to 20
Planning Framework: Philosophic
Unit Length: 2 to 3 weeks
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Description

Thinking about complex numbers is like thinking about five-dimensional objects: they pose a serious challenge to the imagination, and yet they become quite simple matters with the help of a few easy-to-master mathematical tools. In learning about complex numbers, students finally get a complete picture of the full range of numbers, a quest that began around the time they learned to speak, with the teaching of the first few natural numbers. However, to complete this quest, they must put all their faith in the power of their mathematical intelligence, accepting that they can graph and compute numbers that their imagination reels at. In introducing the concept of complex numbers, this unit brings out the magnificent power of abstract thinking.

Unit Outline

1. Identifying powerful underlying ideas

What underlying ideas or theories seem best able to organize the topic into some coherent whole? What are the most powerful, clear, and relevant theories, ideologies, metaphysical schemes, or meta-narratives?

The most powerful underlying idea or theory in this topic:
Despite the name, imaginary numbers baffle the imagination. While it is possible to conceptualize, and even visualize, negative numbers or fractions or irrational numbers, it is very difficult to wrap the mind around the idea of imaginary numbers. Despite this bafflement, imaginary numbers fit quite sensibly into a system of complex numbers, and we hardly need to stretch our prior understanding of arithmetic to accommodate imaginary numbers. The arithmetic of complex numbers is probably the first topic students will come across where the power of their mathematical training will exceed the power of their imagination. The experience of learning about complex numbers reinforces the tremendous power of abstract thinking and the mathematical tools that facilitate it.

An alternative:
Using the Argand diagram to represent complex numbers graphically, we expand the one-dimensional number line of the real numbers into the two-dimensional plane of the complex numbers. By introducing complex numbers, we are not simply adding on an extra piece to students' prior understanding of arithmetic. Rather, we are showing that that prior understanding gave us only a one-dimensional picture of a two-dimensional world. In teaching complex numbers, we can emphasize that the full field of arithmetic is only now heaving into view. If we express complex numbers in the from $z = a + bi$, then all the arithmetic students have learned until now covers only that exceptional set of cases where $b = 0$. This approach may lead us to broach the fundamental theorem of algebra, showing that only with the concept of complex numbers can we settle on the elegance of an algebraically closed number system.

2. Organizing the content into a theoretic structure

Content that exposes the scheme or theory most vividly:

On the whole, most of us toggle between what we might call a "mathematical" and an "intuitive" approach to numbers. An intuitive approach relies heavily on visualization and conceptualization. We can grasp what a negative number is because we can visualize it on a number line or on a Cartesian co-ordinate grid, or because we can conceptualize it in terms of the withdrawals listed on our bank statements. A mathematical approach abandons intuition entirely, and relies exclusively on mathematical tools. We have no intuitive sense of whether our calculation of the product of two five-digit numbers is accurate, but we can be confident that it is to the extent that we can be confident in our facility with the mathematical tools we bring to bear on the problem.

The challenge posed by complex numbers, we can impress upon students, is that intuition is mostly useless. There are intuitive ways of grasping on to negative, or even irrational, numbers, but there really isn't any intuitive conception of the square root of -1. Most of us ease our way into the world of mathematics like acrobats learning new tricks, assured by the presence of the safety net of intuition lest our developing skills should fail us. With the introduction of complex numbers, the safety net is removed, and we must see how far our mathematical skills can carry us unaided.

To the end of encouraging students to trust in their mathematical skills, it might be worthwhile to make explicit the distinction between a mathematical and an intuitive understanding of numbers. We might encourage students to clarify to themselves when and to what extent they rely on the two different kinds of understanding, and to get a sense of the limitations of each.

2.2. Organizing the body of the lesson or unit

What meta-narrative provides a clear overall structure to the lesson or unit?
Lay out the content that will present a strong meta-narrative of the topic:
We could introduce students to the idea of thinking in terms of complex numbers by means of an analogous problem: what does five-dimensional space look like? We could wrestle for hours with the idea of five-dimensional space, struggling to visualize it and waxing philosophical on the question of whether such a thing exists at all. Or we could show that, just as a point in two-dimensional space can be expressed with the co-ordinates (x, y), so a point in five-dimensional space can be expressed with the co-ordinates (v, w, x, y, z). Even though we struggle to visualize and conceptualize it, thinking about five-dimensional space mathematically is as simple as scribbling down a few extra variables.

Imaginary numbers are similar. The square root of -1 boggles the imagination, but incorporating it into our prior understanding of arithmetic is really quite simple.

By clinging only to what our intuition allows, we are confined to a three-dimensional world of real numbers. If we can learn to block out the puzzled complaints of our intuition and learn to appreciate just how much farther we can go with the unaided power of mathematics, we can embrace a much richer and more wonderful world with limitless dimensions and imaginary numbers. Students might respond positively to the romance of the idea that mathematical thinking releases them from the straightjacket of common sense. Where common sense refuses to go, the mathematically trained intellect can venture defiantly simply by scribbling down equations and refusing to be daunted by the fact that they make no intuitive sense.

Students might also take comfort in knowing that they are not alone if they find the idea of complex numbers hard to grasp. A survey of the history of complex numbers shows the strong, and sometimes furious, opposition with which the idea of complex numbers was met. The term "imaginary number" was coined by Descartes, and was meant derogatorily. Many prominent mathematicians refused to accept complex numbers, and they only became widely accepted in the nineteenth century. Oddly enough, despite their obvious disconnection from reality, complex numbers now have a number of practical applications related to electric circuits, quantum mechanics, fluid dynamics, and other fields. In effect, the wheel has come full circle: while we may have no intuitive grasp of complex numbers, an electrical engineer probably does.

3. Developing the tools to analyze the theoretic structure

What mathematical or scientific tools will help us analyze the phenomena touched upon in the general idea or theory? How can we develop and explore these tools in a way that underlines their pertinence to the general idea or theory?

List areas in which students' sense of agency can be engaged and encouraged:
In order to think with complex numbers, students will first need to grasp the idea of imaginary numbers. In a single lesson, we should be able to introduce the square root of -1, i, and show how factors of i can be added, subtracted, multiplied, and divided. Once students have grasped the mechanics of imaginary numbers, we can introduce them to the world of complex numbers of the form $z = a + bi$. In order to reinforce for them just how much more powerful their thinking becomes when they abandon the intuitive world of real numbers, we can show that all the numbers they had worked with up until now were just an exceptional

subset of the complex numbers, where b = 0. By walking students through the simple arithmetic of complex numbers, we can show them how easy it is to manipulate these seemingly baffling numbers.

By introducing Argand diagrams, we render complex numbers easier to visualize. We also provide a demonstration of the power of the abstract thinking opened up by our acceptance of complex numbers: the one-dimensional number line becomes a two-dimensional plane. It is as if we have spent our whole lives just walking in a straight line, looking only directly forward or directly backward, and now we realize that we can look and move sideways as well. Chess players may appreciate the analogy of the real numbers to the restricted movement of a pawn and the complex numbers to the free-ranging movement of a queen.

Depending on how advanced the students are, and how much class time there is, we could also convert the Cartesian coordinates of the Argand diagrams to polar coordinates, and show how each complex number can be represented as a vector with a magnitude and a direction. In exploring these aspects of complex numbers, we can start illustrating the real-world applications of complex numbers.

4. Taking account of the limitations of the theoretic structure

What relevant problems are we not able to solve with the tools we have acquired in this unit? What further complications do these problems add? What sorts of tools might help us to address them

List the main problems raised by the general idea or theory that cannot be solved with the tools at hand, and indicate what further tools might help:

Before emphasizing the limitations of our newly acquired theory, we might want to reflect that, in terms of numbers, we have reached a kind of limit. We might want to give an informal presentation of the Fundamental Theorem of Algebra, showing how, with the complex numbers, we have an algebraically closed number system. While we cannot find a solution to every equation involving whole numbers without referring to numbers outside the set of whole numbers, and while we cannot find a solution to every equation involving real numbers without referring to numbers outside the set of real numbers, we can say with confidence that every equation involving complex numbers will have a solution within the set of complex numbers. Students might be humbled to learn that this very important theorem was first proved in the doctoral thesis of Carl Friedrich Gauss, when the mathematician was only 21 years old.

While the set of complex numbers is algebraically closed, we can warn students that this is not the end of the matter, hinting ominously at the existence of hypercomplex numbers, such as quaternions, octonions, and sedenions, which students will not encounter until well into their university years.

Perhaps more productively, we could point to the riches that await a deeper look into the world of complex numbers. In particular, we could show how expressing complex numbers as vectors in a polar coordinate system opens up an interesting set of problems tying together complex numbers, linear algebra, and trigonometry. If we wish to whet students' appetites, we can present them with the astonishing formula $e\ i = -1$.

5. Encouraging development of students' sense of agency

What features of the knowledge will best allow us to encourage the students' developing sense of agency?

List areas in which students' sense of agency can be engaged and encouraged:
With the introduction of complex numbers, students are asked to leave the real world behind and launch boldly into the world of mathematical abstraction. What they discover is that the "real" world of real numbers is only an infinitesimally small subset of the fantastic world of complex numbers. They also discover that they can freely navigate this fantastic world with the mathematical tools they have acquired, even though their intuition and common sense are halted at the gate. Ideally, this experience can reinforce for students just how powerful abstract, mathematical thinking can be. They have acquired a set of skills that take them far beyond where a rigid, reality-based mindset can go.

The experience of breaking out of the box of intuitive thinking and into the wide world of mathematical thinking might appeal to adolescents who are struggling with restrictions and limitations placed on their own lives. The mathematics of complex numbers presents a classic case of "thinking outside the box," and engenders the sense that traditional limitations can be transcended with sufficient ingenuity and skill. The experience of learning about complex numbers might thus give students a sense of agency toward problems by which they feel boxed in.

6 . Conclusion

How can we ensure that the students have grasped not only a new set of theoretical tools, but also an understanding of where these tools fit into a larger conceptual framework? How can we ensure that the students understand not only how to apply what they have learned, but also why it works and why it should matter that they understand it?

What concluding activity will help both to reinforce the tools that students have learned and to reinforce their place in the larger meta-narrative?

*I*n order to reinforce the idea that, with the complex numbers, we have attained an algebraically closed number system, we might conclude by reviewing the different kinds of numbers students have encountered between infancy and now. By examining, in turn, the natural numbers, the whole numbers, the integers, the rational numbers, the real numbers, and the complex numbers, we can see how each is a subset of the next, and that each, besides the complex numbers, can be used to formulate an equation whose solution requires a broader set of numbers. This review might give students an appropriate sense of closure in knowing that, with the complex numbers, they now know all the numbers there are to know.

7 . Evaluation

How can we know whether the content has been learned and understood, whether students have developed a theory or general idea, elaborated it, and attained some sense of its limitations?

What forms of evaluation will give adequate evidence that the students have learned and understood the content and also have developed and used some theory or abstract idea:
In order to gain the necessary confidence in dealing with complex numbers, students are going to have to do a certain number of repetitive exercises. Similarly, standard quizzes and tests can evaluate whether or not students have gained this confidence. Impressing upon students the power of thinking mathematically is useless if they do not attain a level of comfort at it.

Of course, while attaining the technical ability to work with complex numbers is essential, such technical ability does not guarantee that students appreciate the power of the mathematical tools they have acquired. This appreciation might be evaluated through class discussion and the responses of students to the conceptual material being presented. Alternatively, we could assign students to make presentations on some of the non-technical material in this unit, such as the history of complex numbers, the Fundamental Theorem of Algebra, the concept of an algebraically closed set, or the applications of complex numbers.