Differential Calculus

Topic: Differential Calculus
Target Age: 16 to 99
Planning Framework: Philosophic
Unit Length: 2 to 3 weeks
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Description

In our changing world, calculus is the mathematical tool that formalizes change and expresses it in terms that we can make clear sense of. In theory, any process of change, from the acceleration of a sprinter to the falling of leaves in autumn, can be expressed as a mathematical function. Differential calculus provides the initial set of tools by which we can analyze these functions and develop a sense for the inner workings of processes of change. This unit introduces students to the tools and methods of the differential calculus. The word "calculus" is a Latin word meaning "pebble or stone" (as it is still used in medicine and dentistry), and indicates that pebbles (prehistorically and later) were used to "calculate."

Unit Outline

1. Identifying powerful underlying ideas

What underlying ideas or theories seem best able to organize the topic into some coherent whole? What are the most powerful, clear, and relevant theories, ideologies, metaphysical schemes, or meta-narratives?

The most powerful underlying idea or theory in this topic:
Differential Calculus was the first mathematical tool invented to measure change itself. Until the invention of the calculus in the late seventeenth century, mathematicians had been able to model only static states of affairs, or at best variable rates represented by functions. Calculus is the first tool to take change itself as an object of study, giving change a central place in our understanding of the universe.

An alternative:
“Calculus” is shorthand for “infinitesimal calculus,” because it deals with infinitesimal line segments, which have an infinitely small, but non-zero length.

How can we make sense of the infinitesimal, and how does this struggle clarify for us the concepts of infinity, continuity, and limit?

2. Organizing the content into a theoretic structure

2.1 Initial access

How can the underlying theory or idea be made vivid? What content best exposes it and shows its power to organize the topic?

Content that exposes the scheme or theory most vividly:
We can begin by asking what change in general is and how we register change in its different manifestations. What do putting on a new set of clothes, melting snow, and revising our preconceptions about a person have in common that we can refer to all three of them in terms of a change taking place? A change involves a final state that is different from an initial state, yes, but what sorts of things might we be able to say of this progression from one state to another that might help us to understand it better? How long did it take? How different is the final state from the initial state, and in what way? What unit of measurement, if any, can we use to indicate the change? How might we observe or measure the change? What value is there in measuring change, and what difficulties are there?

One powerful conceptual tool for measuring change that students should already have is the polynomial function. Though a function might not seem the most obvious way of analyzing the process of changing clothes or changing our mind, it can be instructive to examine how these processes can, at least in some sense, be expressed as polynomial functions. It might be a fun challenge to find a way to graph, for example, a person's state of undress as a function of time.

2.2. Organizing the body of the lesson or unit

What meta-narrative provides a clear overall structure to the lesson or unit?

Lay out the content that will present a strong meta-narrative of the topic:

We can appreciate the power of calculus by understanding the revolutionary impact it had on the science of its day, and how it was in many ways the centerpiece in a more general revolution in human beings' understanding of the cosmos and their place in it. The infinitesimal calculus was developed by Newton and Leibniz (the conflicting claims to priority by these two figures, and the ensuing controversy, might also make an interesting topic of discussion) as a means of expressing change in mathematical terms. For the first time, scientists could study change in rigorous and formalized terms, facilitating a whole new view of the universe as dynamic and changing rather than as static and fixed.

We can begin by looking at the medieval conception of the cosmos, with the earth fixed at the center of an ordered and harmonious cosmos. We might consider Aristotle's assumption about inertia, taken for granted for many centuries, that everything will come to rest unless some force is constantly and actively keeping it in motion. That is, Aristotle and his medieval successors saw rest as the natural state of things. We can also look at the various ways in which this conception of a universe at rest influenced the socio-political arrangements of the medieval world. For example, we could show how this conception of a fixed cosmos reflects the fixed social hierarchy of the day, placing kings at the top of a Great Chain of Being that allows for little social mobility.

To better appreciate the historical place of the invention of calculus, we can consider the general change in the West's understanding of the world that came about in the Renaissance, set off by the development of the printing press, the discovery of the Americas, the Protestant Reformation, and so on. These developments set off in turn a series of new developments in the sciences, from Copernicus to Galileo to Kepler to Newton. The world ceased to be a fixed entity at the center of a universe whose natural tendency was toward rest, but rather came to be seen as one moving element in a dynamic and ever-changing cosmos. To make sense of this new cosmos, it was important to develop tools that could measure and quantify its nature. Enter calculus as the knight in shining armor to make sense of our dynamic universe.

3. Developing the tools to analyze the theoretic structure

What mathematical or scientific tools will help us analyze the phenomena touched upon in the general idea or theory? How can we develop and explore these tools in a way that underlines their pertinence to the general idea or theory?

List the major tools necessary for the analysis and explain their pertinence:
Students should already be familiar with polynomial functions, and should know how to graph a curve and how to measure the slope of a line. They should also be familiar with sequences and series, and their attendant limits. Differential calculus is simply a matter of combining these familiar elements in new ways. Students will need to learn how to identify, measure, and express the limits of a function, recognize and plot tangents to curves that show the slope of a curve at a given point, and then to generalize these concepts of limit and slope to calculate the derivative of a polynomial function. Once students have learned the

central concept of computing a derivative, the rest of the material consists simply of elaborations on this central concept. The power rule, the sum rule, the product rule, and the chain rule, as well as formulas for the derivatives of trigonometric functions and other tricky functions, all build on this central concept of the derivative.

While teaching these mathematical tools, it is worth constantly bearing in mind, and reminding the students, of the role these various tools play in the larger narrative of developing a generalized means of analyzing change. We are trying to make the inner workings of change evident, and clear to view. For instance, by drawing a line that is tangent to a curve, we are showing plainly just what the slope of the curve is at that infinitesimal point where curve and tangent meet. We are turning the elusive fluidity of a curve into a simple, straight line. We are finding ways to make complicated things simpler.

4. Taking account of the limitations of the theoretic structure

What relevant problems are we not able to solve with the tools we have acquired in this unit? What further complications do these problems add? What sorts of tools might help us to address them?

List the main problems raised by the general idea or theory that cannot be solved with the tools at hand, and indicate what further tools might help:
Most units on differential calculus include a discussion of the anti-derivative, which leads naturally into integral calculus. Our purpose here is not to teach students integral calculus—that makes for another unit entirely—but to make them aware of the sorts of problems that differential calculus is unable to solve that can be addressed by integral calculus. We may also want to indicate the sorts of problems that could only be solved once students have learned differential equations.

The most obvious point we can make is that we know how to find the slope of a curve, but we do not know how to find the area under a curve. In order to make this lack of knowledge pertinent, it might be useful to give some examples of the kinds of problems that can be solved by determining the area under a curve, and show how this set of problems relates closely to the kinds of problems students have learned to solve with differential calculus. If we have taught students about the anti-derivative, we can show how that tool can serve as a starting point for the more involved methods of integral calculus.

5. Encouraging development of students’ sense of agency

What features of the knowledge will best allow us to encourage the students’ developing sense of agency?

List areas in which students’ sense of agency can be engaged and encouraged:

Learning differential calculus is not exactly a spur to get students writing letters to their local elected official, but it does provide a greater sense of control over a subject that is of very stressful concern to young adults: change. There are a number of ways we can highlight the relevance of their newfound mastery over a very abstract subject to the world around them. As we discussed earlier, change is ubiquitous, and all change can, with greater or lesser relevance, be expressed as a polynomial function. A fun exercise might involve challenging students to find an example of change that is most resistant to graphical representation, and then challenging them to find a means of graphing it. A further exercise might invite students to graph the function of some very non-mathematical kind of change—the brightness of my mood as a function of time over the course of a day, for example—and then to draw the derivative of that graph and try to explain what the derivative represents. The point of these kinds of exercises is primarily to give students a sense of agency with regard to the changes in their lives. While calculus itself may not help them in their romantic troubles, a greater sense of confidence in recognizing and analyzing change just might.

6 . Conclusion

How can we ensure that the student's theories or general ideas are not destroyed but are recognized as having a different status from the facts they are based on? How can we ensure that the decay of belief in the Truth of theories or general ideas does not lead to disillusion and alienation?

What concluding activity will help to both support and show problems with students’ theories, ideas, met-narratives, ideologies, etc.:

This might be a good point to return to our historical narrative about the discovery of calculus by Newton and Leibniz, and its central place in bringing about a new worldview from the old medieval order. We could challenge students to think of problems that cannot be solved without calculus, which highlight the limitations of the worldview in which such problems had to be left unsolved. We could brainstorm a list of the kinds of solutions to problems and the kinds of inventions that would not have been possible without calculus. These exercises might lead into further reflection on how our own assumptions about the world around us are influenced by the mathematical tools our culture makes available to us.

7 . Evaluation

How can we know whether the content has been learned and understood, whether students have developed a theory or general idea, elaborated it, and attained some sense of its limitations?

What forms of evaluation will give adequate evidence that the students have learned and understood the content and also have developed and used some theory or abstract idea:

Unpleasant as it is, repetition is one of the best ways to reinforce a basic command of the mathematical tools without which calculus is nothing more than a nice idea. Though exercises, homework, and quizzes are obviously useful in this regard, we should note that these well-used methods of evaluation only give us a sense of students' mastery of the tools for doing calculus. We need further methods of evaluation to monitor the extent to which students have absorbed these mathematical tools into a broader understanding of where they belong in the students' overall conceptual frameworks. The students' grasp of these broader themes might emerge through exercises normally anathema to a math class, such as essays, class discussion, and open debate. Some of the exercises suggested above on how students can relate their technical mastery of differential calculus to a broader understanding of change might also serve as useful means of evaluating progress.